# MATH 2220 Lecture Notes

Pradhyum Rajasekar

11 Feb 2026 – present

## Contents

**Vectors and Sets** 2

**Paths** 3

**Surfaces** 4

**Differential forms** 7

**Computations** 8

**Sample Proofs** 9

**Linear transformation**

$T : \mathbb{R}^n \to \mathbb{R}^m$ if $T(a\boldsymbol{x} + b\boldsymbol{y}) = aT(\boldsymbol{x}) + bT(\boldsymbol{y})$

$T$ can always be represented as a unique matrix.

**Determinant**

A function from an $n$ by $n$ square matrix to the reals. Properties:

1. $|\det[\boldsymbol{x}_1 \; \boldsymbol{x}_2 \; \ldots]|$ is the volume of the $n$-dimensional parallelepiped spanned by the $\boldsymbol{x}_i$.
2. The determinant is zero if two rows or columns are equal.
3. Swapping two rows inverts the sign of the determinant
4. The determinant is positive if the basis vectors align with the standard basis vectors.
5. $\det[A] = \det[A^T]$
6. $\det[AB] = \det[A] \det[B]$

**Cross product**

Defined by the property, for any three vectors $\boldsymbol{x}$, $\boldsymbol{v}$, $\boldsymbol{w}$:

$$\boldsymbol{x} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \det[\boldsymbol{x} \; \boldsymbol{v} \; \boldsymbol{w}]$$

Can be geometrically interpreted as the volume of a parallelepiped.

**Cauchy-Schwartz inequality**

For any two $\boldsymbol{v}$, $\boldsymbol{w}$:

$$\boldsymbol{v} \cdot \boldsymbol{w} \leq ||\boldsymbol{v}|| \, ||\boldsymbol{w}||$$

**Open ball**

Of radius $r$ centered at $\boldsymbol{a}$:

$$B(\boldsymbol{a}, r) = \{\boldsymbol{x} \in \mathbb{R}^n \; : \; ||\boldsymbol{x} - \boldsymbol{a}|| < r\}$$

**Open set**

A set $U \subseteq \mathbb{R}^n$ such that:

$$\forall \boldsymbol{x} \in U. \; \exists r \in \mathbb{R}. \; B(\boldsymbol{x}, r) \subset U$$

**Closed set**

A set $U \subseteq \mathbb{R}^n$ such that $\mathbb{R}^n - U$ is open.

**Level set**

Of an $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ for a given $c$:

$$\{\boldsymbol{x} \in U \; : \; f(\boldsymbol{x}) = c\}$$

Typically $n - 1$ dimensional subset of $U$.

**Preimage**

Of an $f : A \to B$ and some $S \subseteq B$:

$$f^{-1}(S) = \{x \in A \; : \; f(x) \in S\}$$

**Elementary function**

Any expression built up from addition, subtraction, multiplication, division, exponentiation ($e^x$), and inverses. Elementary functions are continuous on their domain and their partial derivatives are elementary.

The domain of an elementary function depends on its fractional exponents $x^r$:

- if $r = \frac{m}{n}$ where $n$ is even, then the domain is $(0, \infty)$
- if $r = \frac{m}{n}$ where $n$ is odd, then the domain is $\mathbb{R}$ *except* in the case below:
- if $r < 0$, 0 is not in the domain.

**Simply connected set** A set such that (1) there is a smooth path between every pair of points and (2) every closed curve can be continuously contracted to a point.

**Vector field** A function that maps to a vector.

# Paths

**Parameterization** Let $I$ be some interval of the reals. $\alpha : I \to \mathbb{R}^n$ (a **path**) parameterizes the **curve**:

$$C = \{\boldsymbol{x} : \exists t \in I.\ \boldsymbol{x} = \alpha(t)\}$$

**Basic terms** These quantities are calculated fairly simply:

**Velocity** $\boldsymbol{v}(t) = \alpha'(t)$

**Acceleration** $\boldsymbol{a}(t) = \alpha''(t)$

**Speed** $v(t) = ||\boldsymbol{v}(t)||$

**Arclength** $s(t) = \int_a^b v(t)\, dt$

**Frenet vectors** These vectors define an orthonormal basis for any point along a curve:

$$\boldsymbol{T} = \frac{\boldsymbol{v}}{v} \qquad \text{(Unit tangent)}$$

$$\boldsymbol{N} = \frac{\boldsymbol{T}'}{||\boldsymbol{T}'||} \qquad \text{(Principal normal)}$$

$$\boldsymbol{B} = \boldsymbol{T} \times \boldsymbol{N} \qquad \text{(Binormal)}$$

**Curvature** For a given unit tangent vector $\boldsymbol{T}$:

$$\kappa = \frac{||\boldsymbol{T}'||}{v}$$

Measures a normalized rate of change of the tangent vector's direction.

**Torsion** For a given binormal vector $\boldsymbol{B}$, a principal normal vector $\boldsymbol{N}$ and a $c(t)$ for which $\boldsymbol{B}' = c\boldsymbol{N}$:

$$\tau = -\frac{c}{v}$$

Measures a normalized rate of "wobbling" of the plane of path's motion.

**Frenet-Serret formulas** The first and third equations result from definitions. The second one results from the application of the product rule to $\boldsymbol{N}' = (\boldsymbol{B} \times \boldsymbol{T})'$ with the other two:

$$\begin{aligned}
\boldsymbol{T}' &= \phantom{-\kappa v\boldsymbol{T}} \quad \kappa v\boldsymbol{N} \\
\boldsymbol{N}' &= -\kappa v\boldsymbol{T} \qquad\qquad + \tau v\boldsymbol{B} \\
\boldsymbol{B}' &= \phantom{-\kappa v\boldsymbol{T}} \quad - \tau v\boldsymbol{N}
\end{aligned}$$

**$v\kappa\tau$ theorem** Two paths $\alpha : I \to \mathbb{R}^n$ can be translated and rotated to fit on top of each iff they have the same $v(t)$, $\kappa(t)$, and $\tau(t)$.

**Oriented curve** A path associated with a direction.

**Line integral** For a smooth path $\alpha : [a, b] \to U$ traversing an oriented curve $C$ and a vector field $\boldsymbol{F} : U \to \mathbb{R}^n$,

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{s} = \int_a^b \boldsymbol{F}(\alpha(t)) \cdot \alpha'(t)\ dt$$

This is very much like the chain rule. The value of the integral doesn't depend on the parameterization.

**Conservative field** A vector field $\boldsymbol{F}$ such that $\boldsymbol{F} = \nabla f$ for some scalar function $f$. Properties:

· $\int_C \boldsymbol{F} \cdot d\boldsymbol{s} = 0$ along any closed path $C$.
· In $\mathbb{R}^2$, $\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$.
· In $\mathbb{R}^3$, $\nabla \times \boldsymbol{F} = 0$.

If the mixed partials are equal and $\boldsymbol{F}$ is defined on a simply connected set, then $\boldsymbol{F}$ is conservative (converse of the mixed partials theorem).

**Winding number** Let $\boldsymbol{W} = \frac{1}{r^2}(-y, x)$. Note that $(0, 0)$ is not in the domain of $\boldsymbol{W}$. The winding number of an oriented curve $C$ is equal to

$$\frac{1}{2\pi} \int_C \boldsymbol{W} \cdot d\boldsymbol{s}$$

This is equal to how many times the curve goes counterclockwise around the origin.

## Surfaces

**Affine function** Where $T$ is a linear transformation, for some $\boldsymbol{a}$ and $\boldsymbol{b}$, a function $f$ such that:

$$f(\boldsymbol{x}) = \boldsymbol{b} + T(\boldsymbol{x} - \boldsymbol{a})$$

**Equation of a plane** Where $\boldsymbol{n}$ is a vector normal to the surface and $\boldsymbol{a}$ is an offset:

$$\boldsymbol{n} \cdot (\boldsymbol{x} - \boldsymbol{a}) = 0$$

Substituting $\boldsymbol{x} = (x, y, z)$ yields the Cartesian equation.

**Continuity** $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at a point $\boldsymbol{a}$ if:

$$\lim_{\boldsymbol{x} \to \boldsymbol{a}} f(\boldsymbol{x}) = f(\boldsymbol{a})$$

Epsilon-delta definition of continuity:

$$\forall \epsilon.\ \exists \delta.\ f(B(\boldsymbol{a}, \delta)) \subset B(f(\boldsymbol{a}), \epsilon)$$

**Limit** $\lim_{\boldsymbol{x} \to \boldsymbol{a}} f(\boldsymbol{x}) = L$ iff, given some

$$\tilde{f}(\boldsymbol{x}) = \begin{cases} f(\boldsymbol{x}) & \boldsymbol{x} \neq \boldsymbol{a} \\ L & \boldsymbol{x} = \boldsymbol{a} \end{cases}$$

$\tilde{f}$ is continuous at $\boldsymbol{a}$.

**Partial derivative** Given $f : U \to \mathbb{R}$ where $U \subseteq \mathbb{R}^n$, some $\boldsymbol{a}$, and an integer $1 \le j \le n$:

$$\frac{\partial f}{\partial x_j}(\boldsymbol{a}) = \lim_{h \to 0} \frac{f(\boldsymbol{a} + h\boldsymbol{e}_j) - f(\boldsymbol{a})}{h}$$

Partial derivatives at a point can be evaluated by assigning all variables not being differentiated their given values.

**Jacobian matrix** Also called the **derivative**. Of $f : \mathbb{R}^n \to \mathbb{R}^m$ at $\boldsymbol{a}$:

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

**Differentiability** $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a point $\boldsymbol{a}$ if:

$$\lim_{\boldsymbol{x} \to \boldsymbol{a}} \frac{||f(\boldsymbol{x}) - (f(\boldsymbol{a}) + Df(\boldsymbol{x} - \boldsymbol{a}))||}{||\boldsymbol{x} - \boldsymbol{a}||} = 0$$

This derives from the single variable equivalent: $f' = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$
Conditions for differentiability:

- Every partial derivative at the point must exist (necessary).
- Continuous (necessary).
- Is $C^1$ (sufficient).

Is defined for $f : U \to \mathbb{R}$ where $U$ is closed by expanding the domain of $f$ to be open.

**Class $C^k$** A function is $C^k$ if every $k$-th degree partial derivative exists and is continuous (for $C^1$, this is the entries of $Df$). Implies differentiability. $C^\infty$ means every partial derivative of every degree exists and is continuous.

**Gradient** Of an $f : \mathbb{R}^n \to \mathbb{R}$. The vector counterpart to the Jacobian:

$$\boldsymbol{\nabla} f = (Df)^T$$

**Mean value theorem** For $f : \mathbb{R} \to \mathbb{R}$,

$$f(b) - f(a) = f'(c) \cdot (b - a)$$

Can be applied to multiple variables by selecting each coordinate in turn, or by parameterizing $f$:

$$f(\boldsymbol{b}) - f(\boldsymbol{a}) = Df(\boldsymbol{c})(\boldsymbol{b} - \boldsymbol{a})$$

**Little chain rule** Given $f : \mathbb{R}^n \to \mathbb{R}$, $\alpha : I \to \mathbb{R}^n$ where $I$ is an interval of the reals,

$$(f \circ \alpha)'(t) = Df(\alpha(t))\ \alpha'(t)$$

**Extrema** Given $f : U \to \mathbb{R}$ and $\boldsymbol{a} \in \mathbb{R}^n$, $\boldsymbol{a}$ is a local maximum if

$$\exists \epsilon.\ \forall \boldsymbol{x} \in B(\boldsymbol{a}, \epsilon).\ f(\boldsymbol{x}) \le f(\boldsymbol{a})$$

and respectively for minimum. A global maximum is when $\boldsymbol{a}$ is the maximum in the entire domain.

**Hessian matrix**

Given $f : \mathbb{R}^2 \to \mathbb{R}$,

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

The Hessian can be used as a second-order approximation to $f$ at $\boldsymbol{a} \in \mathbb{R}^2$:

$$f(\boldsymbol{x}) = f(\boldsymbol{a}) + Df(\boldsymbol{x}) \cdot (\boldsymbol{x} - \boldsymbol{a}) + \frac{1}{2}(\boldsymbol{x} - \boldsymbol{a})^T \cdot H(\boldsymbol{x}) \cdot (\boldsymbol{x} - \boldsymbol{a})$$

The proof follows from parameterizing $f$ along a line from $\boldsymbol{x}$ to $\boldsymbol{a}$ and simplifying the second-order Taylor expansion for this new function.

**Critical point**

Given $f : \mathbb{R}^n \to \mathbb{R}$ and $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{x}$ is a critical point if

$$(Df)(\boldsymbol{x}) = \begin{bmatrix} 0 & 0 & \ldots & 0 \end{bmatrix}$$

All extrema are critical points. If $H(\boldsymbol{x}) \neq 0$, then $\boldsymbol{x}$ is a **nondegenerate** critical point. If $H(\boldsymbol{x}) > 0$, then if $\frac{\partial^2 f}{\partial x^2}$ is positive, then $\boldsymbol{x}$ is a minimum, otherwise it is a maximum. If $H(\boldsymbol{x}) < 0$, then $\boldsymbol{x}$ is a saddle point.

**Integrability**

The **integral** of a function $f : R \to \mathbb{R}$, $R = [a, b] \times [c, d]$ is defined as

$$\iint_R f(x, y)\ dA = \lim_{\substack{\Delta x_i \to 0 \\ \Delta y_i \to 0}} \sum_{i,j} f(\boldsymbol{p}_{ij}) \Delta x_i \Delta y_i$$

If the limit exists, then the function $f$ is integrable.

Conditions for integrability:

- Bounded, and discontinuous at at most a finite number of points or curves. (sufficient)

**Parameterization**

Surfaces can be parametrized with some function $\sigma : D \to \mathbb{R}^n$ where $D$ is closed, $\sigma(s, t) = (x(s, t), y(s, t), z(s, t))$.

Common parameterizations:

**Polar coordinates** $\sigma(r, \theta) = (r \cos\theta, r \sin\theta)$.

**Cylindrical coordinates** $\sigma(r, \theta, z) = (r \cos\theta, r \sin\theta, z)$.

**Spherical coordinates** $\sigma(\rho, \phi, \theta) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)$. Note that $\phi$ is the angle of the point from the $z$-axis.

**Boundary**

The boundary $\partial D$ of a 2D surface $D$ is the curve oriented such that $D$ is to your left as you traverse it (compare with the Frenet-Serret vectors).

The boundary of a surface in 3D is the set of all points don't look Euclidean locally (e.g. it just "stops"): for a hemisphere, it is the base; for a cylinder, it is the top and bottom; for a sphere, there is no boundary. The orientation is the same as in 2D, except that the upwards direction is defined to be the orienting normal $\boldsymbol{n}$.

The boundary of a volume is oriented such that the normal vector points away from the volume.

**Green's theorem**

In $\mathbb{R}^2$, given a vector field $\boldsymbol{F}$ and a region $D$,

$$\int_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{s} = \iint_D \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)\ dx\ dy$$

Compare with Stoke's theorem restricted to 2D. Note that $\boldsymbol{F}$ has to be defined over $D$: compare with the integration of $\boldsymbol{W}$.

**Orientation**

A field $\boldsymbol{n}$ defined for some surface $S$ such that, for every point on $S$, $\boldsymbol{n}$ is a unit normal vector. *Outward* and *inward* normals refer to such functions that point outwards or inwards respective to the origin, respectively.

Given a parameterization $\sigma : D \to \mathbb{R}^3$, $D \subseteq \mathbb{R}^2$, note that a natural choice of an orientation is the normalization of $\frac{\partial\sigma}{\partial s} \times \frac{\partial\sigma}{\partial t}$; if $\boldsymbol{n}$ is equal to this, then $\boldsymbol{n}$ *preserves orientation*; otherwise, if it is the negation, then $\boldsymbol{n}$ *reverses orientation.*

**Surface integral**

Is used to calculate flux. Given a vector field $\boldsymbol{F}$ and a surface $S$ parameterized by $\sigma : D \to \mathbb{R}^3$, then

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \pm \iint_D \boldsymbol{F}(\sigma(s,t)) \cdot \left(\frac{\partial\sigma}{\partial s} \times \frac{\partial\sigma}{\partial t}\right)\ ds\ dt$$

where the $\pm$ resolves to $+$ if $\sigma$ is orientation-preserving; otherwise, it is a $-$. Compare this to an integral with respect to surface area.

**Stokes's theorem**

Given a surface $S$ and a vector field $\boldsymbol{F}$,

$$\int_{\partial S} F \cdot d\boldsymbol{s} = \iint_S (\nabla \times F) \cdot d\boldsymbol{S}$$

**Curl field**

A vector field $\boldsymbol{F}$ such that $\boldsymbol{F} = \nabla \times \boldsymbol{G}$. By Stokes's theorem, the surface integral of a curl field only depends on the boundary of the surface. The divergence of a curl field is zero.

**Gauss's theorem**

Given a volume $W$ and a vector field $\boldsymbol{F}$,

$$\iint_{\partial W} \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_W (\nabla \cdot W)\ dV$$

# Differential forms

**Differential form**

A thing that can be integrated (such as $dx$). A *0-form* is a point, a *1-form* is a linear integrand, a *2-form* is integrated with a double integral, and so forth. These all have orientation.

**Wedge product**

An antisymmetric, associative operation that constructs "volumes" out of other differential forms. The wedge product of a differential form with itself (e.g. $dx \wedge dx$) is 0.

**Integration**

The integral of a differential form simplifies to its standard form:

$$\int_D f(x,y)\ dx \wedge dy = \int_D f(x,y)\ dx\ dy$$

**Substitution**

To substitute for a differential form, follow this example:

$$\begin{aligned} dx \wedge dy &= \det \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix}\ ds\ dt \\ &= \det\left( \begin{bmatrix} \frac{\partial}{\partial s} \\ \frac{\partial}{\partial t} \end{bmatrix} \otimes \begin{bmatrix} x \\ y \end{bmatrix} \right)\ ds\ dt \end{aligned}$$

where $\otimes$ represents the outer product, and the first vector in that product is an operator.

Then, when integrating a differential form, turn it into its pullback form by substituting using the substitution above, then factoring out the $dsdt$. This can then be integrated over the new domain.

**Exterior derivative** Notated as $d$. Rules:

1. For a 0-form $f$: $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$
2. It distributes across addition.
3. When applied to a scalar $f$ multiplied by a differential form $\eta$: $d(f\eta) = df \wedge \eta$, where $df$ can be expanded with the first rule.

**Fundamental theorem of calculus** When notated using differential forms, Green's theorem, Stokes's theorem, and Gauss's theorem become the same theorem: given a manifold $M$ and a differential form $\omega$,

$$\int_M d\omega = \int_{\partial M} \omega$$

## Computations

**Integral of $\sin^2$** Where $n$ is a multiple of $\frac{1}{2}$,

$$\int_0^{n\pi} \sin^2\theta \; d\theta = \int_0^{n\pi} \cos^2\theta \; d\theta = \frac{n\pi}{2}$$

**Chain Rule** Given $f : \mathbb{R}^m \to \mathbb{R}^n$ and $g : \mathbb{R}^q \to \mathbb{R}^m$,

$$D(f \circ g)(\boldsymbol{x}) = Df(g(x))Dg(x)$$

Note that $Df(g(x))$ is an $n \times m$ matrix and $Dg(x)$ is an $m \times q$ matrix; therefore, the matrix product is a $n \times q$ matrix.

**Directional derivative** For an $f : \mathbb{R}^n \to \mathbb{R}$ and a unit vector $\boldsymbol{u}$ at some $\boldsymbol{a}$,

$$\boldsymbol{u} \cdot \boldsymbol{\nabla} f(\boldsymbol{a})$$

Derives applying the chain rule on $f(\alpha(t))$, $\alpha(t) = \boldsymbol{u}t + \boldsymbol{a}$

**Normal vector** Given $f : \mathbb{R}^n \to \mathbb{R}$ and some $\boldsymbol{a} \in \mathbb{R}^n$ and a level surface defined as $f(\boldsymbol{x}) = 0$, the vector normal to $f$ at $\boldsymbol{x}$ is computed by:

$$(\boldsymbol{\nabla} f)(\boldsymbol{a})$$

**Continuous preimage rule** Given a continuous $f : U \to \mathbb{R}^m$ and a set $S \subseteq \mathbb{R}^m$, if $S$ is open, $f^{-1}(S)$ is open.

**Lagrangian multiplier** When optimizing some function $f : \mathbb{R}^n \to \mathbb{R}$ subject to some constraint $g : \mathbb{R}^n \to \mathbb{R}$, $g(\boldsymbol{x}) = c$, then, at the optimal point $\boldsymbol{x}$,

$$\boldsymbol{\nabla} f(\boldsymbol{x}) = \lambda \boldsymbol{\nabla} g(\boldsymbol{x})$$

**Centroid** Of some set $D \subseteq \mathbb{R}^n$:

$$\overline{x_i} = c_i = \frac{1}{\text{Vol}(D)} \iint \cdots \int_D x_i \, dV$$

**Surface area** An integral $\iint_S f \, dS$ with respect to surface area is evaluated by parameterizing the surface with $\sigma$ and using this definition:

$$dS = \left|\left| \frac{\partial \sigma}{\partial s} \times \frac{\partial \sigma}{\partial t} \right|\right| \, ds \, dt$$

**Change of variable** Given an injective transformation $T : W^* \to W$ for which $T(W^*) = W$, $W, W^* \subseteq \mathbb{R}^n$,

$$\iint \cdots \int_W f(\boldsymbol{x}) \, dx_1 \ldots \, dx_n = \iint \cdots \int_{W^*} f(T(\boldsymbol{u})) |\det DT(\boldsymbol{u})| \, du_1 \ldots \, du_n$$

# Sample Proofs

---

**Proposition 1.** *The set of points in $\mathbb{R}^2$ such that $x^2 + y^2 = 1$ is a closed set.*

*Proof.* Consider the complement $\overline{S}$ of this set. For an arbitrary element $\boldsymbol{x} = (a, b)$, if $a^2 + b^2 < 1$, let $\delta = 1 - ||\boldsymbol{x}||$; if $a^2 + b^2 > 1$, let $\delta = ||\boldsymbol{x}|| - 1$. Consider $S_b = B(\boldsymbol{x}, \delta)$: in the former case, for some $\boldsymbol{v} \in S_b$, $||\boldsymbol{v}|| < ||\boldsymbol{x}|| + \delta = 1$ by the triangle inequality; in the latter case, $||\boldsymbol{v}|| > ||\boldsymbol{x}|| - \delta = 1$. This verifies that $\overline{S}$ is open, and therefore, that $S$ is closed. □

**Proposition 2.** *The following function $f$ is not continuous at $(0,0)$ regardless of what value is assigned to $f(0,0)$:*

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

*Proof.* Note that $\lim_{x \to 0} f(x, 0) = \frac{x^2}{x^2} = 1$ and that $\lim_{y \to 0} f(0, y) = -\frac{y^2}{y^2} = -1$. Consider $\epsilon = \frac{1}{2}$. There is no value of $f(\boldsymbol{x})$ for which $B(f(\boldsymbol{x}), \epsilon)$ contains both 1 and -1. However, the limits demonstrate that there are points arbitrarily close to $(0,0)$ that map to 1 and -1; therefore, $f(B(\boldsymbol{x}, \delta))$ will contain 1 and -1 for any $\delta > 0$. This demonstrates that the epsilon-delta criterion of continuity is false for this function at $(0,0)$, regardless of what the value of $f(\boldsymbol{x})$ is. □

**Proposition 3.** *The following function $f$ is continuous at $(0,0)$:*

$$f(x, y) = \begin{cases} \frac{x^3 + 2y^3}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

*Proof.* Note that:

$$\begin{aligned} \frac{x^3 + 2y^3}{x^2 + y^2} &= \frac{r^3 \cos^3 \theta + 2r^3 \sin^3 \theta}{r^2} \\ &= r \cos^3 \theta + 2r \sin^3 \theta \\ &\leq 3r \end{aligned}$$

since cos and sin is always bounded above by 1. Then, for any $\epsilon > 0$, let $\delta = \frac{\epsilon}{3}$: then, $f(B((0,0), \delta)) \subset B(f((0,0)), \epsilon)$. □

**Proposition 4.** *The following function $f$ is not differentiable at $(0,0)$:*

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

*Proof.* First, let us calculate $Df$ at $(0,0)$:

$$\begin{aligned} Df(0,0) &= \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}\Big|_{(0,0)} \\ &= \begin{bmatrix} 0 & 0 \end{bmatrix} \end{aligned}$$

Now, let us plug it into the definition of differentiability:

$$\begin{aligned} \lim_{\boldsymbol{x} \to (0,0)} \frac{f(\boldsymbol{x}) - Df(\boldsymbol{x})}{||\boldsymbol{x}||} &= \frac{xy/(x^2+y^2)}{\sqrt{x^2+y^2}} \\ &= \frac{xy}{(x^2+y^2)^{3/2}} \end{aligned}$$

Let us approach the origin on the line $y = x$:

$$\begin{aligned} \frac{xy}{(x^2+y^2)^{3/2}} &= \frac{x^2}{2\sqrt{2}|x|^3} \\ &= \frac{2^{-3/2}}{|x|} \end{aligned}$$

Since this value diverges from 0 as the $x$ approaches 0 and therefore the origin, this function is not differentiable at $(0,0)$. ☐

**Proposition 5.** *Differentiability implies continuity.*

*Proof.* For some function $f$, differentiability means:

$$\begin{aligned} \lim_{\boldsymbol{x} \to \boldsymbol{a}} \frac{f(\boldsymbol{x}) - (f(\boldsymbol{a}) + (Df)(\boldsymbol{x} - \boldsymbol{a}))}{||\boldsymbol{x} - \boldsymbol{a}||} &= 0 \\ \lim_{\boldsymbol{x} \to \boldsymbol{a}} f(\boldsymbol{x}) - (f(\boldsymbol{a}) + (Df)(\boldsymbol{x} - \boldsymbol{a})) &= 0 \\ \lim_{\boldsymbol{x} \to \boldsymbol{a}} f(\boldsymbol{x}) - \lim_{\boldsymbol{x} \to \boldsymbol{a}} (Df)(\boldsymbol{x} - \boldsymbol{a}) &= f(\boldsymbol{a}) \\ \lim_{\boldsymbol{x} \to \boldsymbol{a}} f(\boldsymbol{x}) &= f(\boldsymbol{a}) \end{aligned}$$

which is the definition of continuity. ☐

**Proposition 6.** $\boldsymbol{F}(x,y) = (y, -x)$ *is not conservative.*

*Proof.* Take the integral of $\boldsymbol{F}$ around the unit circle $C$ clockwise:

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{s}$$

Since the integrand is always positive and greater than zero, the value of this integral is not zero, which contradicts the definition of a conservative field. ☐