MATH 1120 Lecture Notes

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1 Integration

1.1 Integration by Substitution

FTC I implies that we can evaluate $\int_a^b f(x)\,dx$ with the antiderivative of $f(x)$.

u-substitution derives from reversing the chain rule:

$$\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$$
$$\int f'(g(x)) \cdot g'(x) = f(g(x)) + C$$

and the u in this case is $g(x)$. In practice, we look for a function-derivative pair.

$\int f(x)\,dx$ is the *general antiderivative*, the family of all antiderivatives of $f(x)$. It is called the *indefinite integral*.

1.2 Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

Select a u that can be easily differentiated and a v that can be easily integrated; example:

$$\begin{aligned}\int x \ln x \; dx &= \frac{1}{2}x^2 \cdot \ln x - \int \frac{1}{2}x^2 \cdot \frac{1}{x} \; dx \\ &= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\,dx \\ &= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C\end{aligned}$$

Note that here, differentiating ln allows for the powers to cancel out! Don't just go for the power rule differentiation.

Integration by parts can be used for deriving integrals for inverse functions; it seems to be very similar to an area-based formula; here's an example for arctan:

$$\begin{aligned}\int \arctan(x)\,dx &= x\arctan(x) - \int \frac{x}{1+x^2}\,dx \\ &= x\arctan(x) - \frac{1}{2}\int \frac{1}{1+u}\,du \\ &= x\arctan(x) - \frac{1}{2}\ln(|1+u|) + C \\ &= x\arctan(x) - \frac{\ln(|1+x^2|)}{2} + C\end{aligned}$$

1.3 Integration with Trigonometry

For expressions involving sines and cosines, use a u-substitution. Isolate a trigonometric function by itself to do the substitution. Example:

$$\begin{aligned}\int \sin^4(x)\cos^5(x) \; dx &= \int \sin^4(x)\left(1-\sin^2(x)\right)^2 \cos(x) \; dx \\ &= \int u^4\left(1-u^2\right)^2 \; du\end{aligned}$$

which can be easily solved using the power rule. This works when at least one power is odd.

When both powers are even, use these identities:

$$\begin{aligned}\cos^2 &= \frac{1}{2}(1+\cos(2x))\\ \sin^2 &= \frac{1}{2}(1-\cos(2x))\\ \sin(x)\cos(x) &= \frac{1}{2}\sin(2x)\end{aligned}$$

Similarly, for expressions involving tangents and secants, use a u-substitution, with $u = \tan(x), du = \sec^2(x)dx$, or $u = \sec(x), du = \tan(x)\sec(x)dx$. The tangent substitution only works when the power on the secant is even; the secant substitution only works when the power on the tangent is odd and at least 1 secant. Such expressions rely on the Pythagorean identity with tan and sec:

$$\tan^2(x) + 1 = \sec^2(x)$$

This rule works pretty much the same for cot and csc, except that $\cot'(x) = -\csc^2(x)$.

When integrating a product of two trigonometric functions with different arguments, use the *product-to-sum* identities:

$$\cos(x)\cos(y) = \frac{1}{2}(\cos(x+y)+\cos(x-y))$$

1.3.1 Trig Substitution

Note that for expressions such as

$$\int x\sqrt{25x^2-4}\,dx$$

, a trig substitution is not needed; only when there is no du term does trig substitution need to be used. Here is a table of expressions in the integrand and what trig substitution to use:

Expression	Trig substitution
$\sqrt{b^2x^2-a^2}$	$x = \frac{a}{b}\sec\theta$
$\sqrt{b^2x^2+a^2}$	$x = \frac{a}{b}\tan\theta$
$\sqrt{a^2-b^2x^2}$	$x = \frac{a}{b}\sin\theta$

Note that csc, cot, and cos can be used for all of these with (maybe?) slight variation.

When doing this process, put an absolute value operator after simplifying the square root! However, **in an indefinite integral**, you can assume that it is positive, but **note it** so that when doing bounded integrals, you are aware of it: in bounded integrals, it can lead to sign errors. Just an example of what I mean:

$$\begin{aligned}2\sqrt{\sec^2\theta-1} &= 2\sqrt{\tan^2\theta}\\ &= 2|\tan\theta|\end{aligned}$$

For $\sec\theta$, $\theta\in[0,\pi], \theta\neq\frac{\pi}{2}$ for $\sqrt{\sec^2\theta-1}$ to be positive; in the other case, when $\theta\in[\pi,2\pi]$, the same expression would be $-\tan\theta$. Be wary of this, especially in definite integrals where this can cause a sign error.

1.4 Partial Fraction Decomposition

For each power of a term in the denominator, there is a fraction. Example:

$$\frac{x^3+x+7}{(x-2)^3(x+1)^2(2x-7)} = \frac{A}{2x-7}+\frac{B}{x+1}+\frac{C}{(x+1)^2}+\frac{D}{x-2}+\frac{E}{(x-2)^2}+\frac{F}{(x-2)^3}$$

If the numerator has a higher degree than the denominator, do polynomial long division then do PFD on the remainder portion. For each quadratic factor in the denominator, write a term like so:

$$\frac{Ax + B}{x^2 + 1}$$

where the denominator is any quadratic factor; repeated quadratic factors look the same as repeating linear factors.

Example:

$$\begin{aligned}
\frac{x^3 + 10x^2 + 3x + 36}{(x-1)(x^2+4)^2} &= \frac{A}{x-1} + \frac{Bx + C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2} \\
x^3 + 10x^2 + 3x + 36 &= A(x^2+4)^2 + (Bx+C)(x-1)(x^2+4) + (Dx+E)(x-1) \\
50 &= 25A \implies A = 2 \\
&\text{And so on.}
\end{aligned}$$

These huge systems can be solved by plugging in enough x values to turn it into a linear system; alternatively, they can be solved by expanding the RHS, but this only works well with small powers.

If solving using x values, select values that take advantage of symmetries and that turn into small coefficients, preferably 1; in exam questions, there is usually such a value.

1.5 Numeric Integration

L_n denotes the left Riemann sum:

$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + ... + f(x_{n-1})\Delta x$$

R_n denotes the right Riemann sum:

$$R_n = f(x_1)\Delta x + ... + f(x_n)\Delta x$$

.

M_n denotes the middle Riemann sum:

$$M_n = f(\overline{x_1})\Delta x + ... + f(\overline{x_n})$$

where $\overline{x_i}$ is $\frac{x_i + x_{i-1}}{2}$. This is not equal to the trapezoidal sum.

T_n denotes the trapezoidal sum:

$$T_n = D_1 + D_2 + ...D_n$$

where $D_i = \frac{f(x_{i-1}) + f(x_i)}{2}\Delta x$. The trapezoidal sum can be rewritten as

$$\frac{1}{2}(L_n + R_n)$$

Another representation of midpoint summation (middle Riemann sum): draw a tangent at the midpoint and draw a trapezoid such that the sides are the bounding x-values. Using this, it is simple to see that:

- For concave up: midpoint underestimates, trapezoidal overestimates
- For concave down: midpoint overestimates, trapezoidal underestimates

Simpson's rule:

$$\begin{aligned}
\int_a^b f(x)\ dx &= \frac{T_n + 2M_n}{3} \\
&= S_{2n}
\end{aligned}$$

This approximation relies on the fact that $\frac{T_n}{M_n} \approx -2$. The $2n$ is due to the fact that this formula uses both endpoints of the integrated areas and midpoints of those segments. God I am so tired.

Simpson's rule gives a perfect approximation for any cubic or lower.

2 Applications of Integration

Obviously, where $f(x)$ is the top and $g(x)$ is the bottom, the area between them is simply the difference between the integrals between the bounds; the bounds are usually the graphs' intersections or something like that.

You can also integrate horizontally (again, self-evident).

2.1 Arc Length

$$\text{Arc Length} = \int \sqrt{1 + (y')^2}\ dx$$

This falls out of the Pythagorean theorem.

2.2 Solids of Revolution

If you're using disks, each disk would have the volume

$$\pi y^2\ dx$$

which can be integrated over the relevant values to calculate volumes.

If you're using shells?, each piece would have the volume

$$2\pi y\ dx$$

which can be integrated over the relevant range.

2.3 Mass and Center of Mass

The total mass of some linear object with mass density function $\rho(x)$ in units of grams per centimeter or whatever is simply the integral of the object over its length bounds.

The center of mass is slightly more complex:

$$\bar{x} = \frac{1}{M} \int_{x_1}^{x_2} x\rho(x)\ dx$$

which is basically a weighted average of every position on the rod. So really pretty simple. An easy way to check this is to check units.

2.4 Improper Integral

A *proper integral* is one where both bounds are finite and the function being integrated is bounded on that interval. An *improper integral* is an integral where this is not the case.

For infinite bounds, it should be calculated as a limit:

$$\begin{aligned}\int_0^\infty e^{-x}\ dx &= \lim_{b\to\infty} \int_0^b e^{-x}\ dx \\ &= \lim_{b\to\infty} [-e^{-b} + e^0]\ dx \\ &= 0 + 1 = 1\end{aligned}$$

Note that the limit should be evaluated last.

Note: when calculating an integral from negative to positive infinity, don't evaluate it as the limit from $-a$ to a; instead, split it into two integrals:

$$\int_{-\infty}^\infty f(x)\ dx = \int_{-\infty}^0 f(x)\ dx + \int_0^\infty f(x)\ dx$$

When dealing with a finite bound where $f(x)$ is unbounded, take the limit of the problematic bound:

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{b\to 0^+}\int_b^1 \frac{1}{x^2}\,dx$$

Note the directional nature of the limit: that is the main difference between these two cases.

If integrating across an asymptote, split it into two integrals with an endpoint on that asymptote.

2.4.1 Convergence

To determine whether an integral converges or not, a particularly useful method is to compare it to $\frac{1}{x^P}$: integrals of the form

$$\int_1^{-\infty} \frac{1}{x^P}\,dx$$

converge for $P > 1$ and diverge for $0 < P \leq 1$. Inversely, integrals of the form

$$\int_0^1 \frac{1}{x^P}\,dx$$

converge for $0 < P \leq 1$ and diverge for $P > 1$; these both should be intuitively obvious from graphs and the knowledge that the integral of $\frac{1}{x}$ diverges.

2.5 Physics

There are two main kinds of physics problems: ones that deal with work and ones that deal with pressure. The work ones are fairly intuitive (for me, at least): they are governed by the equation $W = Fs$, and you essentially integrate across s as the variable changes. This is often confounded with some elements of time, but once a relationship between F and s is found, it reduces to be pretty simple.

The unintuitive one for me are questions that deal with pressure. Consider a problem where you are given a box with dimensions, X, Y, and Z, with Z pointing up, and you are asked to find the force of water on the X side. Here's how you do it:

1. Find the pressure at a given depth. This is usually: $P = \rho hg$. The unit of water that you are considering must reciprocate this pressure in all directions; therefore, the pressure on the wall is also ρhg.

2. Find a differential area on the wall: $dA = X\ dh$ (some problems will have an area that varies with height).

3. $F = PA$. Multiply the two quantities and integrate:

$$F = \int_0^Z \rho hg \cdot X\ dh$$

I think once you think about it a little, it should make sense.

Always be careful about signs, especially when adding a quantity to some fixed point to get a shifted point.

3 Differential Equations

Definition: an equation that involves a function and one or more of its derivatives.

Example: let $P(t)$ represent the population of a certain country which increases at a rate proportional to its population. Then, this can be represented as the equation:

$$\frac{d}{dt}P(t) = kP(t)$$

The order of a differential equation is the highest order of its terms.

An *autonomous equation* has form

$$\frac{dy}{dx} = f(y)$$

An *equilibrium solution* is one where y is constant; they can be found for an autonomous equation by simply finding the roots of $f(y)$. Autonomous solutions can be categorized into *stable*, *unstable*, or *semistable*:

- A stable solution is one where nearby solutions approach the equilibrium solution as $x \to \infty$.
- An unstable solution is the opposite: where nearby solutions move away as x increases.
- A semistable solution is where it is stable in some regions and unstable in others. (there are usually only two regions)

These types of equilibria can be distinguished by analyzing the sign of f' around the equilibrium: if a solution has negative derivatives above it and positive derivatives below it, it is stable; reversed, it is unstable; if the derivatives have the same sign, it is semistable.

Solutions cannot cross equilibria since the derivative on an equilibrium line is 0.

A *separable* differential equation is one where

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

This kind of equation can easily be solved by cross-multiplying and integrating both sides. This creates a constant of integration on one side:

$$\int g(y)\ dy = \int f(x)\ dx + C$$

Note that all autonomous equations are seperable equations, with the function $f(x) = 1$ being divided by the function $g(y) = \frac{1}{h(y)}$ where $h(y)$ was the RHS of the original autonomous equation.

The constant of integration **must** be added in at the step where integration is done; this is because, as it propagates through subsequent computations, it may become a coefficient or become embedded inside another function.

3.1 Models

For all types of models, make sure to keep track of units! That will be a lifesaver, and is a good way of verifying the validity of your work quickly.

3.1.1 Inflow-Outflow

In these questions, you are given some sort of container containing some amount A of substance, and some substance is flowing in, and some substance is flowing out. Usually, the amount or rate of substance flowing in is given as some I (usually constant) and the outflow rate is proportional to the concentration of substance in the container.

A common class of problems deals with concentration: there is a tank with volume V and with concencentration C, a inflow-outflow rate r, where the inflow concentration is some constant I and the outflow concentration is the concentration of the tank. This can be modeled and solved as such:

$$\begin{aligned}
V\frac{dC}{dt} &= Ir - Cr \\
\frac{V}{I-C}\ dC &= r\ dt \\
-V\ln(I-C) &= rt + D \\
I - C &= Ae^{-rt/V} \\
C &= I - Ae^{-rt/V}
\end{aligned}$$

Given some initial concentration C_0,

$$C_0 = I - A$$
$$A = I - C_0$$

The reason I solve this equation here is because it pops up a lot: know the general form of solutions to common differential equations.

Note that there are many similar concepts. In particular, be aware of the difference between concentration and amount, and which one your differential equation solves for!

3.1.2 Logistic equation

The rate of population growth can be assumed to be proportional to the population:

$$\frac{dP}{dt} = kP$$

k can be approximated as $\frac{P'(0)}{P(0)}$, where $P'(0)$ is the slope of P between -1 and 1. The core assumption is essentially the same as saying that the per capita growth rate is constant:

$$\frac{\frac{dP}{dt}}{P} = k$$

However, this is often not the case: the per capita growth rate is usually a linear function of population. This assumption leads to the differential equation:

$$\frac{dP}{dt} = kP(N - P)$$

This is the *logistic equation*. It is used to model scenarios where there is a limit on resources: Earth, animals on an island, etc.

This differential equation has two equilibrium solutions, one at $P = 0$ and another at $P = N$. Therefore, it is obvious that N is the maximum (stable) limit: it is called the *carrying capacity*.

An analytic formula can be derived:

$$\begin{aligned}
\frac{dP}{dt} &= kP(N - P) \\
\frac{1}{P(N-P)}\ dP &= k\ dt \\
\int \frac{1}{N}\left(\frac{1}{P} + \frac{1}{N-P}\right)\ dP &= kt + C \\
\ln P - \ln(N - P) &= Nkt + C \\
\frac{P}{N-P} &= Ae^{Nkt} \\
P(Ae^{Nkt} + 1) &= Ae^{Nkt}N \\
P &= \frac{N \cdot Ae^{Nkt}}{Ae^{Nkt} + 1} \\
&= \frac{N}{1 + Ae^{-Nkt}}
\end{aligned}$$

This is the most general form; now, if an initial population P_0 is given:

$$Ae^{Nkt} = A = \frac{P_0}{N - P_0}$$

This is the solution to the logistic equation.

4 Series

4.1 Taylor Series

An n-th degree Taylor series $T_f(x)$ of a function $f(x)$ is a polynomial such that the k-th derivatives at a certain point a always match for every $k < n$. This can be easily formalized as

$$T_f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} x^i$$

where $f^{(k)}(a)$ refers to the k-th derivative of a function (with $k = 0$ indicating the function itself).

4.2 Geometric Series

As you no doubt know, a geometric series S has first term a_0 and common ratio r:

$$S_n = a_0 + a_0 r + a_0 r^2 + ... + a_0 r^n$$

Finite geometric series can be evaluated as thus:

$$\begin{aligned}
S_n &= a_0 + a_0 r + a_0 r^2 + ... + a_0 r^n \\
rS_n &= a_0 r + a_0 r^2 + ... + a_0 r^n + a_0 r^{n+1} \\
S_n(r-1) &= a_0 r^{n+1} - a_0 \\
S_n &= a_0 \frac{r^{n+1} - 1}{r - 1} \\
&= a_0 \frac{1 - r^{n+1}}{1 - r}
\end{aligned}$$

In words, this can be remembered as the difference between the term after the last one and first term, divided by the difference between the ratio and 1.

Infinite geometric series can be evaluated as a special case of the formula above: when $|r| < 1$, $\lim_{n\to\infty} r^n$ converges to 0, so:

$$S_\infty = \frac{a_0}{1 - r}$$

4.3 Convergence Testing

A common type of question is calculating the values of x for which a given series converges. These tests let you determine the radius or interval of convergence, or whether a series converges at all.

4.3.1 Limit Tests

Let there be some series T_i. Then,

$$\begin{aligned}
T_i \text{ converges} &\implies \lim_{i\to\infty} T_i = 0 \\
\lim_{i\to\infty} T_i \neq 0 &\implies T_i \text{ diverges}
\end{aligned}$$

The first is a necessary but not sufficient condition for convergence. The latter can be used to show that a series diverges.

4.3.2 Integral Test

Some series T_i is convergent or divergent if and only if its integral is the same. Mathematically, if $T_i = f(i)$,

$$\sum_{i=k}^{\infty} T_i \text{ converges } \iff \int_k^{\infty} f(x)\, dx \text{ converges}$$

From this and the fact that $\frac{1}{x^P}$ converges for $P > 1$ follows that the *p-series* is convergent when $P > 1$:

$$\sum_{n=k}^{\infty} \frac{1}{x^P}$$

4.3.3 Comparison Test

Given some series T_i, let there be some other series A_i; then, if $T_i \leq A_i$ and A_i converes, T_i converges. If $T_i \geq A_i$ and A_i diverges, then T_i diverges.

Another kind of comparison test is the limit comparison test, which applies if $T_i \geq 0$ and $A_i > 0$: if

$$\lim_{i\to\infty} \frac{A_i}{T_i} = c$$

where c is some finite, positive number, then either both series converge, or diverge.

4.3.4 Ratio Test

Say you have a series

$$\sum_{i=0}^{\infty} T_i(x)$$

This series converges if $\frac{T_{i+1}(x)}{T_i(x)} < 1$, diverges if that ratio is greater than 1, and is inconclusive when it is equal to one.

Use this test primarily to find the radius of convergence; this test is inconclusive at the bounds of the interval of convergence.

4.3.5 Alternating Series Test

If the sign of a series T_i alternates with every term, then it converges if $\lim_{i\to\infty} T_i = 0$.

4.4 Alternative Ways of Finding Taylor Series

4.4.1 Substitution

First, here are some common Taylor series:

$$\begin{aligned}
\frac{1}{1-x} &= 1 + x + x^2 + \dots \\
\ln(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots \\
\sin(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots \\
\cos(x) &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \dots \\
e^x &= 1 + x + \frac{1}{2!}x^2 + \dots
\end{aligned}$$

The first two equations have an interval of convergence $|x| < 1$ (ln still holds at $x = 1$) while the rest are valid over all x.

Now, when asked to find the Taylor series for a complex expression, instead find a composition of functions $f(g(x))$ such that $f(x)$ has some known Taylor expansion. Example:

$$\begin{aligned}\frac{2}{1+(2x)^2} &= 2 \cdot \frac{1}{1-(-4x^2)} \\ &= 2 \cdot \left(\sum_{i=0}^{\infty}(-4x^2)^i\right) \\ &= \sum_{i=0}^{\infty} 2(-4)^i x^{2i}\end{aligned}$$

The interval of convergence can be calculated by substituting $g(x)$ into the equation for the interval of convergence of the original equation:

$$\begin{aligned}|-4x^2| &< 1 \\ x^2 &< \frac{1}{4} \\ |x| &< \frac{1}{2}\end{aligned}$$

4.4.2 Differentiation and Integration

If it is known that some function $f(x)$ is either the integral or derivative of some other function $g(x)$, then the Taylor series of f can be calculated by simply taking the derivative or integral of the Taylor series term by term. Very simple (except this is what you messed up on the case exam...).

Note that when the Taylor series is of the form

$$\sum_{k=0}^{\infty} T_k x^k$$

the derivative will be of the form

$$\sum_{k=1}^{\infty} kT_k x^{k-1}$$

(note that k starts from 1 in the latter).

Note that when applying integration, *there will be a constant of integration.* Make sure to plug in a point (usually the point that the series is centered at) to calculate the constant.

4.4.3 With alternative centers

When using alternative centers (that is, $a \neq 0$), then find the form of the x term and manipulate a known series to yield that term. Example: finding the Taylor series of $\frac{1}{x}$ at $a = 1$:

$$\begin{aligned}\frac{1}{x} &= \frac{1}{1-(1-x)} \\ &= \frac{1}{1-(-(x-1))}\end{aligned}$$

And since the Taylor series is centered at 1, the terms would be of the form $a_k(x-1)^k$; therefore, just using the known expansion of this series, we obtain the Taylor series.

4.5 Lagrange Error Formula

A simple non-mathematical statement of the error of a finite Taylor series is as follows: with an alternating series, the error of a sum S_n is less than the next element not included in the sum. There is a nice visual explanation of this fact.

There is a more mathematical and more accurate way of computing the error of a finite series: given a function f with a nth degree Taylor series T_n centered at a, the *Lagrange error formula* states that

$$|f(c) - T_n(c)| \leq K \frac{|c-a|^{n+1}}{(n+1)!}$$

where the LHS is the error at some point c and $f^{(n+1)}(x) \leq K$ for all $x \in [a, c]$.